STOCHASTIC VARIATIONAL APPROACH
TO MINIMUM UNCERTAINTY STATES

Short title: STOCHASTIC VARIATIONAL
APPROACH TO MINIMUM UNCERTAINTY

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Abstract

We introduce a new variational characterization of Gaussian diffusion processes as minimum uncertainty states. We then define a variational method constrained by kinematics of diffusions and Schrödinger dynamics to seek states of local minimum uncertainty for general non-harmonic potentials.

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1. Introduction

Nelson stochastic quantization, originally proposed in 1966 [1], is currently recognized as an independent and self-consistent formulation of quantum mechanics in the language of stochastic processes.

The original Nelson scheme, containing some *ad hoc* assumptions, has been later reconsidered and conceptually clarified in the classic work by Guerra and Morato [2]. They proved that quantum dynamics can be obtained by a stochastic variational principle, i.e. by suitably extremizing the classical action along diffusive trajectories replacing the deterministic ones. In this way stochastic variational methods have creeped in into quantum physics. Both the practical and the conceptual advantages of the variational scheme have since been exploited in a number of different contexts [3], [4], [5].

In this work we introduce a variational method that yields a new characterization of Gaussian diffusion processes. The analysis is carried out for the one-dimensional case, but

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the results can be easily extended in any number of dimensions. We prove that a necessary and sufficient condition for a diffusion process $q(t)$ to be Gaussian is that a certain positive-defined functional of the probability density $\rho(x, t)$ acquires a non-zero global minimum at each fixed instant of time. This functional is built by considering the variances Δq^2 of $q(t)$ and Δu^2 of the logarithmic derivative $u(x, t) = \nu \partial_x [\ln \rho(x, t)]$, where ν is the diffusion coefficient. We then consider the "uncertainty" product $\Delta q^2 \Delta u^2$ as the quantity to be extremized against smooth variations $\delta \rho$ of the probability density.

For the Nelson diffusions of stochastic mechanics the quantity $u(x, t)$ is the so-called osmotic velocity, and $\Delta q^2 \Delta u^2$ is the osmotic uncertainty product. One can then rephrase our result by saying that Nelson Gaussian diffusions are all and only those that minimize the osmotic uncertainty product at each fixed instant of time.

Historically, an uncertainty principle involving the diffusion coefficient had been already conjectured by Fürth in the study of Brownian Motion [6], and explicitly derived for the Nelson diffusions of stochastic mechanics by exploiting Schwartz's inequality [7], [8]. Saturation of the osmotic uncertainty then yields the standard harmonic-oscillator coherent and squeezed states [9]. Obviously, the same states can also be recovered in the stochastic variational approach.

In fact, they are identified in two steps. Knowledge of the Gaussian density allows first to derive their phase. This is achieved by imposing Fokker-Planck kinematics through the continuity equation. The latter connects the density with the gradient of the phase, i.e. the so-called current velocity v. The harmonic-oscillator potential is then determined by means of Schrödinger dynamics in the Hamilton-Jacobi-Madelung hydrodynamic form. Thus, in this scheme, the kinematical and dynamical inputs are exploited *a posteriori*, once the osmotic uncertainty functional has been minimized.

On the other hand, in stochastic mechanics there is a current uncertainty term $\Delta q^2 \Delta v^2$ that has to be added to the osmotic one to yield the full Heisenberg uncertainty product $\Delta \hat{q}^2 \Delta \hat{p}^2$. Minimization of the osmotic term alone takes care of the current uncertainty by fixing it to be either zero (Heisenberg minimum) or related to the time-variation of the wave packet spreading Δq (Schrödinger minimum).

It is however possible to consider a variational extremization for the sum of both parts, which is in fact a functional of the density and the current velocity. The latter are not independent but are related by Fokker-Planck kinematics and Schrödinger dynamics, so that minimization can be correctly performed only by including them *a priori* as constraints in the variational procedure. Furthermore, the osmotic uncertainty product $\Delta q^2 \Delta u^2$ is a quantity defined at a fixed instant of time. Upon integration over an arbitrary time-interval $[t_1, t_2]$, it is mapped into a dynamical functional of the process (intuitively, a sort of uncertainty "action"). The correct procedure to minimize this dynamical osmotic uncertainty is then again to impose the kinematical and dynamical constraints.

In the present work we first develop the free variational minimization of the osmotic uncertainty, a result that holds for any diffusion process. We then carry out the study for the various constrained variational schemes.

It is clear that the constrained approach to the minimum osmotic uncertainty, by restricting the class of allowed variations, should in principle yield a larger class of solutions.

In particular, it should allow to assess the existence and to determine the relative minima of the Heisenberg uncertainty product as functions of the assigned external potentials. Unfortunately, due to the intrinsic mathematical complications, we are at present able to carry out the analysis and present a complete solution only for the kinematically constrained problem.

2. Variational characterization of Gaussian diffusion processes

Consider a one-dimensional diffusion process $q(t)$ with the associated normalized probability density $\rho(x,t)$ and a diffusion coefficient ν which may be constant or time-dependent. Introduce the quantity

$$u(x,t) \ \equiv \ \nu\partial_x[\ln\rho(x,t)] \ = \ \nu\frac{\partial_x\rho(x,t)}{\rho(x,t)}\,, \tag{1}$$

and consider the following functional of $\rho(x,t)$:

$$F[\rho] \ \equiv \ \Delta q^2\,\Delta u^2\,, \tag{2}$$

where $\Delta q^2 \equiv Var\{q\} = \langle q^2\rangle - \langle q\rangle^2$, and $\Delta u^2 \equiv Var\{u\} = \langle u^2\rangle - \langle u\rangle^2$ are the variances of $q(t)$ and $u(x,t)$ as functions of time. Here and in the following $\langle\,\cdot\,\rangle$ denotes expectation with respect to the density $\rho(x,t)$: for $u(x,t)$ as function of the process,

$$\langle u\rangle \ = \ \nu\int_{-\infty}^{+\infty}\partial_x\rho(x,t)dx \ = \ 0\,, \tag{3}$$

since the density vanishes at infinity. Thus $u(x,t)$ always has, by construction, zero expectation value, and Δu^2 is simply $\langle u^2\rangle$.

We observe that F is a smooth, positive-defined functional of the density ρ. Pictorially, Δq^2 and Δu^2 are in competition: the first quantity measures the width of the probability density, while the second measures its sharpeness; we therefore expect on intuitive grounds that there exists an absolute minimum of F. To verify this assertion and to determine the explicit minimum value of the functional we extremize F under smooth variations $\delta\rho$ of the probability density; they are taken to be vanishing at the boundaries of integration.

Since minimization of F must preserve normalization of probability, the variations will be constrained by the requirement that at each fixed instant of time $\int\rho(x,t)dx = 1$. We thus consider as the functional to be varied

$$\mathcal{F}[\rho] \ \equiv \ F[\rho] + \lambda(t)\int_{-\infty}^{+\infty}\rho(x,t)dx\,. \tag{4}$$

The Lagrange multiplier $\lambda(t)$ shall be determined *a posteriori* by taking the expectation of the variational equation. The latter is obtained by imposing the vanishing of the variation $\delta\mathcal{F}$ of the uncertainty functional:

$$\delta\mathcal{F}[\rho] \ \equiv \ \delta F[\rho] + \lambda(t)\int_{-\infty}^{+\infty}\delta\rho(x,t)dx \ = \ 0\,. \tag{5}$$

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Long but straightforward calculations lead to the following variational equation for $u(x,t)$:

$$\Delta u^2 \left(x^2 - 2\langle q \rangle x\right) - \left[u^2(x,t) + 2\nu \partial_x u(x,t)\right] \Delta q^2 + \lambda(t) = 0 \,. \tag{6}$$

Taking the expectation of both sides one finds

$$\lambda(t) = \Delta u^2 [\langle q \rangle^2 - 2\Delta q^2] \,. \tag{7}$$

Replacing the above expression for λ in eq.(6) gives the following inhomogeneous nonlinear differential equation of the Riccati type for $u(x,t)$:

$$\Delta q^2 \left[u^2(x,t) + 2\nu \partial_x u(x,t)\right] = \Delta u^2 \, x^2 - 2\Delta u^2 \, \langle q \rangle x + \Delta u^2 [\langle q \rangle^2 - 2\Delta q^2] \,. \tag{8}$$

The right hand-side is of the form $\alpha(t)x^2 + \beta(t)x + \gamma(t)$: it follows that the solution $u(x,t)$, if it exists, must be of the form $a(t)x + b(t)$. The condition that $u(x,t)$ must have a vanishing expectation fixes $b(t) = -a(t)\langle q \rangle$, so that the solution is finally of the form

$$u(x,t) \;=\; a(t)[x - \langle q \rangle] \,. \tag{9}$$

The coefficient $a(t)$ is determined by the requirement that u satisfies identically the variational equation; inserting expression (9) in eq.(8) and equating, respectively, the coefficients of the powers x^2, x and x^0, yields $a(t) = -\Delta u/\Delta q$ and

$$\Delta q^2 \, \Delta u^2 \;=\; \nu^2 \,. \tag{10}$$

From the explicit form

$$u(x,t) \;=\; -\frac{\nu}{\Delta q^2} \left(x - \langle q \rangle\right) \,, \tag{11}$$

we readily trace back the expression for the normalized probability density

$$\rho(x,t) \;=\; \frac{1}{(2\pi)^{1/2}\Delta q}\exp\left\{-\frac{(x - \langle q \rangle)^2}{2\Delta q^2}\right\} \,. \tag{12}$$

It is possible to check, by convexity of the functional or by taking second variations, that the extremum (10) is indeed a minimum.

In conclusion, we have proven the following

Theorem: A necessary and sufficient condition for a smooth probability density $\rho(x,t)$ to be Gaussian is that it makes stationary the functional $Var\{q\} \cdot Var\{u\}$, where $u = \nu \partial_x[\ln \rho(x,t)]$ and ν is an arbitrary function of time (for diffusion processes, the diffusion coefficient). The extremum corresponds to a minimum with value ν^2.

Sufficiency was just shown; necessity trivially follows from the Gaussian characterization, eqs.(11)-(12), and from the definition of $F[\rho]$.

3. Minimum uncertainty Nelson diffusions: coherent states

The theorem proved in the previous section finds application in the framework of stochastic formulation of quantum mechanics. We recall that Nelson stochastic quantization [10] associates to each quantum state of a point particle of mass m a configurational diffusion process $q(t)$ governed by Ito's stochastic differential equation

$$dq(t) \; = \; v_{(+)}(q(t),t)dt + \left(\frac{\hbar}{2m}\right)^{\frac{1}{2}} dw(t), \quad dt > 0 \; . \tag{13}$$

Here $v_{(+)}(q(t),t)$ is the forward drift, $\hbar$ is Planck constant, $\nu = \hbar/2m$ is the diffusion coefficient, and $dw(t)$ is the time-increment of a Wiener process $w(t)$, with expectation $\langle dw(t) \rangle = 0$ and covariance $\langle dw^2(t) \rangle = 2\,dt$. The forward and backward drifts, $v_{(+)}(x,t)$ and $v_{(-)}(x,t)$, are defined by the conditional expectations

$$v_{(+)}(x,t) \; \equiv \; \lim_{\Delta t \to 0^+} \left\langle \frac{q(t+\Delta t) - q(t)}{\Delta t} \Big| q(t) = x \right\rangle,$$

$$\tag{14}$$

$$v_{(-)}(x,t) \; \equiv \; \lim_{\Delta t \to 0^+} \left\langle \frac{q(t) - q(t-\Delta t)}{\Delta t} \Big| q(t) = x \right\rangle.$$

They represent respectively the mean forward (backward) velocity fields.

In the hydrodynamic picture of the process, the drifts are replaced by the osmotic velocity $u(x,t)$,

$$u(x,t) \; \equiv \; \frac{v_{(+)}(x,t) - v_{(-)}(x,t)}{2} \; = \; \frac{\hbar}{2m}\partial_x[\ln \rho(x,t)] \; . \tag{15}$$

and by the current velocity $v(x,t)$,

$$v(x,t) \; \equiv \; \frac{v_{(+)}(x,t) + v_{(-)}(x,t)}{2} \; . \tag{16}$$

Finally, Fokker-Planck equation for the probability density $\rho(x,t)$ takes the form of the continuity equation

$$\partial_t \rho(x,t) \; = \; -\partial_x[\rho(x,t)v(x,t)] \; . \tag{17}$$

At the dynamical level to each single-particle quantum state $\Psi(x,t)$ written in the hydrodynamic form

$$\Psi(x,t) \; = \; \sqrt{\rho(x,t)} \exp\left[\frac{i}{\hbar}S(x,t)\right] \; , \tag{18}$$

there corresponds in stochastic mechanics the diffusion process $q(t)$ with

$$\rho(x,t) \; = \; |\Psi(x,t)|^2 \; , \tag{19}$$

and

$$v(x,t) \; = \; \frac{1}{m}\partial_x S(x,t) \; . \tag{20}$$

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The complex Schrödinger equation with potential $V(x,t)$ for the wave function Ψ is then equivalent to two coupled real equations for the probability density ρ and for the phase S (or, alternatively, for the osmotic and current velocities u and v). They are, respectively, the continuity equation (17) and the gradient of the Hamilton-Jacobi-Madelung (HJM) equation which, in terms of u and v, reads

$$m\partial_t v + mv\partial_x v - mu\partial_x u - \frac{\hbar}{2}\partial_x^2 u = -\partial_x V(x,t) . \tag{21}$$

The correspondence between expectations and uncertainties defined in the stochastic and in the canonic formulations of quantum mechanics are [8]

$$\langle \hat{q} \rangle_\Psi = \langle q \rangle , \quad \langle \hat{p} \rangle_\Psi = m\langle v \rangle ,$$

$$\Delta \hat{q} = \Delta q , \tag{22}$$

$$\Delta \hat{p}^2 = m^2[\Delta u^2 + \Delta v^2] ,$$

where $\hat{q}$ and $\hat{p}$ are the position and momentum operators in the Schrödinger picture, $\langle \cdot \rangle_\Psi$ denote the expectations of the operators in the given state Ψ, $\langle \cdot \rangle$ denote the expectations of the stochastic variables in the state $\{\rho, v\}$ in the Nelson picture, and $\Delta(\cdot)$ denote the variances.

From eq.(22) it follows that in the Nelson picture the Heisenberg uncertainty product $\Delta \hat{q}^2 \Delta \hat{p}^2 \equiv H[\rho, v]$, a functional of the density ρ and of the current velocity v:

$$H[\rho, v] = m^2 \Delta q^2 \Delta u^2 + m^2 \Delta q^2 \Delta v^2 . \tag{23}$$

Apart from the constant factor m^2, the first term in the right hand side of eq.(23) is the osmotic uncertainty functional $F[\rho]$ introduced in section 2, while the second term $m^2 \Delta q^2 \Delta v^2$ is the current uncertainty functional.

We now have the choice to either extremize only the osmotic term $F[\rho]$ under variations $\delta\rho$, or the total uncertainty functional $H[\rho, v]$ under variations $\delta\rho$ and δv. In the former case, one obtains the Gaussian structure (10)-(12), and determines the current velocity through the continuity equation (17). The result is

$$v(x,t) = \langle v \rangle + [x - \langle q \rangle]\frac{d\ln\Delta q}{dt} . \tag{24}$$

If we take Δq to be constant in eq.(24), we have $v = \langle v \rangle$, and $\Delta v = 0$. The quantum uncertainty then collapses to the minimum osmotic uncertainty $\hbar^2/4$, i.e. to the standard absolute Heisenberg minimum. The corresponding states have Gaussian density (12) and phase exponent of the form $\langle \hat{p} \rangle x$: they are the harmonic-oscillator coherent and squeezed states, the potential being determined via the HJM equation (21).

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With Δq time-dependent we have that $\Delta v^2 = (d\Delta q/dt)^2$. It is straightforward to show that it is the Schrödinger position-momentum correlation:

$$\frac{d}{dt}\Delta q \;=\; \frac{1}{m\Delta q}\left[\langle\hat{q}\,\hat{p}\rangle_{symm} \;-\; \langle\hat{q}\rangle\langle\hat{p}\rangle\right], \tag{25}$$

where $\langle\hat{q}\,\hat{p}\rangle_{symm} \;\equiv\; \langle\hat{q}\,\hat{p} + \hat{p}\,\hat{q}\rangle/2$. We thus have

$$\Delta\hat{q}^2\Delta\hat{p}^2 \;=\; \frac{\hbar^2}{4} \;+\; \left[\langle\hat{q}\,\hat{p}\rangle_{symm} \;-\; \langle\hat{q}\rangle\langle\hat{p}\rangle\right]^2. \tag{26}$$

From the above expressions we see that minimum osmotic uncertainty with time-dependent Δq yields the Schrödinger minimum of the Heisenberg uncertainty product [11]. For the detailed structure of coherent and squeezed states in stochastic mechanics we refer to the work by De Martino *et al.* [9].

4. Kinematical constraint and absolute minimum uncertainty

So far, it has been shown that minimizing the osmotic uncertainty functional $F[\rho]$ without constraints reproduces the whole minimum-uncertainty structure of quantum mechanics provided one exploits, after minimization has been carried out, the kinematics of diffusions and the Schrödinger dynamics of quantum states.

On the other hand, a fundamental aspect of the calculus of variations regards the implementation and the meaning of constraints. Minimizing a free functional and then requiring compatibility of the obtained results with some constituent equations is in general not equivalent to minimizing the same functional with the prescribed equations added as constraints. True, it is meaningless to seek a constrained minimization of the osmotic uncertainty $F[\rho]$, since it is a quantity defined and evaluated at each fixed instant of time. However, wishing to implement the kinematical and dynamical equations as constraints, one has simply to replace $F[\rho]$ with its time-integral over an arbitrarily fixed interval $[t_1, t_2]$; this new functional is now dependent on the dynamics: it gives the time-evolution of the osmotic uncertainty along the process, and it is then sound to seek its constrained minimization.

We begin by showing that the crucial aspects of uncertainty stem from the osmotic contribution alone. To this end, we first minimize the total uncertainty functional $H[\rho, v]$. In this case the need for constrained minimization is imposed *ab initio*. Namely, ρ and v are linked by the continuity and HJM equations, so that it is impossible to consider independent variations $\delta\rho$ and δv.

In all generality, one should add both the continuity and the HJM equations to H and minimize the resulting constrained functional. However, we start by attacking the simpler problem of minimizing H with the kinematical constraint alone. In this way dynamics is exploited after minimization, to select the quantum mechanical processes that are compatible with the variational results.

Again, when dealing with time-evolution equations, the quantities to be minimized are not the original functionals, but their time-integrals over an arbitrary interval. Keeping as

before the normalization constraint, one has to consider the following functional of $\rho(x, t)$ and $v(x, t)$:

$$\mathcal{H}[\rho, v] \equiv \int_{t_1}^{t_2} dt\, m^2 \Delta q^2 \left(\Delta u^2 + \Delta v^2\right) + \int_{t_1}^{t_2} dt \int_{-\infty}^{+\infty} dx \left\{ \lambda_0(t)\rho + \lambda_1(x, t)[\partial_t \rho + \partial_x(\rho v)] \right\}, \quad (27)$$

with the arbitrary functions $\lambda_0(t)$ and $\lambda_1(x, t)$ as Lagrange multipliers.

By requiring stationarity of $\mathcal{H}$ with respect to independent variations of ρ and v we are left with the following equations:

$$2\, m^2 \Delta q^2 (v - \langle v \rangle) - \partial_x \lambda_1 = 0\,, \quad (28)$$

$$m^2 (\Delta u^2 + \Delta v^2)(x^2 - 2\langle q \rangle x) + m^2 \Delta q^2 \left(v^2 - 2\langle v \rangle v - \frac{\hbar^2}{m^2} \frac{\partial_x^2 \sqrt{\rho}}{\sqrt{\rho}} \right) + \lambda_0(t) - \partial_t \lambda_1 - v \partial_x \lambda_1 = 0\,. \quad (29)$$

The unknown multipliers λ_0 and λ_1 can be eliminated from these relations by inserting eq.(28) and its time derivative into the space derivative of eq.(29). Differentiation with respect to time obviously yields two distinct results according to whether the spreading Δq is taken to be constant or time-dependent.

In the first case one has, recalling the definition of u, eq.(6),

$$\partial_t(v - \langle v \rangle) + v \partial_x v - \frac{\Delta u^2 + \Delta v^2}{\Delta q^2}(x - \langle q \rangle) + \frac{1}{2} \partial_x \left(u^2 + \frac{\hbar}{m} \partial_x u \right) = 0\,. \quad (30)$$

For the Nelson diffusion of quantum mechanics the above equation must hold together with the HJM dynamics, eq.(21). One can show, after some manipulations, that a sufficient condition for the two equations to be consistent is $v = \langle v \rangle$. This condition reduces eq.(30) to the form

$$\partial_x \left\{ \Delta u^2(x^2 - 2\langle q \rangle x) - \left(u^2 + \frac{\hbar}{m} \partial_x u \right) \Delta q^2 + \lambda(t) \right\} = 0\,. \quad (31)$$

This is just the space derivative of eq.(6) of section 2, with Δq now fixed to be constant, and we again recover the Heisenberg minimum uncertainty states.

In the case of time-dependent Δq, one has instead, from eqs.(28)-(29),

$$\partial_t(v - \langle v \rangle) + 2(v - \langle v \rangle)\, \partial_t \ln \Delta q\; + \; v \partial_x v - \frac{\Delta u^2 + \Delta v^2}{\Delta u^2}(x - \langle q \rangle)$$
$$+ \; \frac{1}{2} \partial_x \left(u^2 + \frac{\hbar}{m} \partial_x u \right) = 0\,. \quad (32)$$

The problem is now slightly more complicated, but one can convince himself, proceeding as before, that HJM dynamics is compatible with the choice of the linear v, eq.(24). By inserting this form of v in eq.(32), the linear u of minimum uncertainty is recovered if the time-evolution of Δq is ruled by

$$\frac{d^2}{dt^2} \Delta q + \frac{1}{\Delta q} \left(\frac{d}{dt} \Delta q \right)^2 = 0\,. \quad (33)$$

One can realize by direct computation that the above expresses the condition for the current uncertainty functional $\Delta q^2 \Delta v^2$, namely the Schrödinger part of the uncertainty product, to be a constant of the motion. Eq.(33) can be explicitly solved to give

$$\Delta q(t) \;=\; \Delta q_0 \sqrt{1 + \beta t}\,, \tag{34}$$

where $\Delta q_0 = \Delta q(t=0)$, $\beta = 2\,(d\Delta q/dt)_{t=0}/\Delta q(0)$; consequently,

$$m^2 \Delta q^2 \Delta u^2 + m^2 \Delta q^2 \Delta v^2 \;=\; \frac{\hbar^2}{4} + m^2 \big[\Delta q^2 (d\Delta q/dt)^2 \big]_{t=0}\,. \tag{35}$$

The corresponding normalized solutions are states of time independent Schrödinger minimum. They are characterized by a wave packet width that evolves according to the diffusive dispersion law Eq.(34).

We see that constrained minimization of the total uncertainty functional selects all the states of Heisenberg minimum uncertainty, but only the special class of constant Schrödinger minima. This is in contrast with the results obtained extremizing the free osmotic functional which allowed to derive all the states of minimum quantum uncertainty. For this reason, we consider again the osmotic uncertainty and seek for its constrained minimization. Proceeding as before, we introduce the functional

$$
\begin{aligned}
\mathcal{C}[\rho, v] \;\equiv\;& \int_{t_1}^{t_2} dt\, m^2 \Delta q^2 \Delta u^2 \\
&+ \int_{t_1}^{t_2} dt \int_{-\infty}^{+\infty} dx \Big\{ \lambda_0(t)\rho + \lambda_1(x,t)\left[\partial_t \rho + \partial_x(\rho v) \right] \Big\}\,.
\end{aligned} \tag{36}
$$

Performing the independent variations of ρ and v and requiring stationarity of $\mathcal{C}$ leads to a couple of equations that replace eqs.(28)-(29)

$$\partial_x \lambda_1 \;=\; 0\,, \tag{37}$$

$$m^2 \Delta u^2 (x^2 - 2\langle q \rangle x) - \hbar^2 \Delta q^2 \frac{\partial_x^2 \sqrt{\rho}}{\sqrt{\rho}} + \lambda_0(t) - \partial_t \lambda_1 - v \partial_x \lambda_1 \;=\; 0\,. \tag{38}$$

Eliminating the unknown multiplier λ_1 from the equations one is left with

$$\partial_x \left\{ \Delta u^2 (x^2 - 2\langle q \rangle x) - \left(u^2 + \frac{\hbar}{m} \partial_x u \right) \Delta q^2 + \lambda(t) \right\} \;=\; 0\,. \tag{39}$$

We are again in the general case of section 2, eq.(6). We remark that now, contrary to the situation faced in the derivation of eqs.(30)-(31), no hypotheses need to be made on the time dependence of Δq. In conclusion, we recover all the states of Heisenberg and Schrödinger minimum uncertainty. Thus, the complete structure of minimum quantum uncertainty is derived by considering the osmotic contribution as the correct functional to be extremized.

5. Discussion and outlook

The analysis carried out in the previous sections implies two main considerations. The first one is the emergence of the crucial role played by the osmotic uncertainty product in determining the fundamental features of the quantum mechanical uncertainty.

The other main implication is that the variational approach lends itself naturally to a further generalization. Namely, it would be very interesting to exploit the dynamics directly as a constraint in the variational procedure to minimize the osmotic uncertainty. The physical motivation for such a construction is that it might allow to discuss the behaviour and the structure of quantum uncertainty for general non-harmonic systems. For instance, it would be interesting to verify whether the Heisenberg uncertainty product exhibits local minima depending on the choice of the potential.

To implement the suggested generalization one should consider the functional

$$
\begin{aligned}
\overline{\mathcal{C}}[\rho, v] \;\equiv\;& \int_{t_1}^{t_2} dt\, m^2 \Delta q^2 \Delta u^2 \\
+\;& \int_{t_1}^{t_2} dt \int_{-\infty}^{+\infty} dx \Big\{ \lambda_0(t)\rho + \lambda_1(x,t)\left[\partial_t \rho + \partial_x(\rho v)\right] \\
+\;& \lambda_2(x,t)\left[\partial_t v + v\partial_x v - \frac{\hbar^2}{2m^2}\partial_x \frac{\partial_x^2 \sqrt{\rho}}{\sqrt{\rho}} + \frac{\partial_x V}{m}\right] \Big\}.
\end{aligned}
\tag{40}
$$

The above defines the most general osmotic uncertainty functional in quantum mechanics, including the normalization, the kinematical and the dynamical constraints. Extremizing it as usual against smooth independent variations of ρ and v one obtains, as in the kinematically constrained scheme, two coupled equations. However, in this case the difficulty of identifying and/or eliminating the unknown Lagrange multipliers has prevented so far an explicit exact solution of the problem. Work is in progress, to obtain both numerical and analytical approximated solutions, and we will report on it elsewhere [12].

Acknowledgement

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